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ON A CLASS OF PLANE AND AXIALLY SYMMETRIC PROBLEMS IN ELASTICITY*

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12. Abstract: In this report we present a unified approach to a class of boundary value problems in plane and axially symmetric elasticity. The formulation given here reduces the problems under consideration to classical problems in potential theory and thus avoids certain technical difficulties which sometimes arise when other methods of solution are used.

1. Introduction: We present in this paper a method for treating certain plane strain (or plane stress) and axially symmetric problems for mediums which are either isotropic or possess a certain type of anisotropy. The method depends on a formulation which permits a uniform treatment of both the plane and axially symmetric problems. Since Airy [1] introduced the biharmonic stress function almost a century ago numerous methods have been proposed for handling plane strain (or stress) problems in elasticity. This Airy's function is still the starting point in the integral transform treatment of such problems (see Sneddon [27]). Formulations based on the theory of functions of a complex variable have been introduced and developed by Kolosoff [15], Muskhelishvilil6 , Stevenson [29], Green [12] and others. These formulations and that of Sen [22] do not involve the stress function. All of the previously mentioned methods require some modification if the region is orthotropic.

It is also known (see [27]) that in corresponding three-dimensional axially symmetric problems the complete stress distribution may be determined from a biharmonic function. This function is again the starting point in integral transform treatment of certain classes of such problems. (see Sneddon [27] and Elliott [6]). Other methods have been proposed by Green [9, vol. 45] and Payne [18] if the medium is isotropic and by Shield [25] and Payne [19] if the medium possesses transverse isotropy.

We are concerned in this paper with plane and axially symmetric problems in which certain stresses and/or displacements

(or combinations of the two) are given either on the boundary of a semi-infinite region or on the two boundaries of an infinite strip. In addition we consider cases in which isolated singularities act at an interior point in the strip or semi-infinite medium.

In our formulation we choose a decomposition of the biharmonic function which reduces the problems in question to classical problems in potential theory. Such a decomposition in the axially symmetric case was given by Payne [18, 19] in his treatment of the crack and punch problems. There are a number of advantages to such a formulation. In the first place it permits a unified treatment of the class of problems under consideration, since the plane and axially symmetric problems are reduced to equivalent boundary value problems in two and three dimensional potential theory and the anisotropic problems under consideration may be reduced to corresponding isotropic problems. In addition such a formulation permits the convenient introduction of curvilinear coordinates, thus avoiding certain technical difficulties which may arise when other methods are used.

Because of the equivalence of the various problems we shall treat as examples in subsequent sections only plane problems. Using the formulation given in [18 , 19] one can, by the same methods as those indicated here, treat the corresponding axially symmetric problems.

There is of course a tremendous literature on problems of this type. We mention in particular the work of Sen [21-24],

Sneddon [26-28], Green [8-12], Elliott [6], Payne [18, 19] Paria [17] and many others [3-5, 7, 13, 14, 20, 25, 30]. Additional references may be found in the bibliography.

2. Elastic medium: In terms of the Airy function χ the stresses and displacement in an isotropic elastic medium are defined as follows:

$$
\begin{aligned}
\tau_{xx} &= \frac{\partial^2 \chi}{\partial y^2} \\
\tau_{yy} &= \frac{\partial^2 \chi}{\partial x^2} \\
\tau_{xy} &= -\frac{\partial^2 \chi}{\partial x \partial y} \\
\frac{\partial u}{\partial x} &= \frac{1}{2\mu}\left[\tau_{xx} - \sigma(\tau_{xx} + \tau_{yy})\right] \\
\frac{\partial v}{\partial y} &= \frac{1}{2\mu}\left[\tau_{yy} - \sigma(\tau_{xx} + \tau_{yy})\right] \\
\tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) &= \frac{1}{2\mu}\tau_{xy}
\end{aligned}
\tag{2.1}
$$

where

$$
\Delta^2 \chi = 0 , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \tag{2.2}
$$

σ is Poisson's ratio, μ a Lame constant and u and v are the displacements in the x and y directions respectively.

We are concerned in this report with problems in which boundary data is given on a straight line boundary which we take as the y-axis in the xy-plane. The elastic medium occupies the region $0 < x < d , -\infty < y < \infty$. In half plane problems d is infinity. Since $x = 0$ is a line on which boundary data is given it is natural to decompose χ in the following way:

$$\chi = \varphi_1' - x\frac{\partial \varphi_2'}{\partial x} \tag{2.3}$$

Where φ_1' and φ_2' are harmonic. Such a decomposition is known as an Almansi decomposition [2], and it is well known that any biharmonic problem can be decomposed in this way. The stresses and displacements are then given by:

$$\begin{aligned}
\hat{\tau}_{xx} &= \frac{\partial^2 \varphi_1'}{\partial y^2} - x\frac{\partial^3 \varphi_2'}{\partial x \partial y^2} = -\left[\frac{\partial^2 \varphi_1'}{\partial x^2} - x\frac{\partial^3 \varphi_2'}{\partial x^3}\right] \\
\hat{\tau}_{yy} &= \frac{\partial^2 \varphi_1'}{\partial x^2} - 2\frac{\partial^2 \varphi_2'}{\partial x^2} - x\frac{\partial^3 \varphi_2'}{\partial x^3} \\
\hat{\tau}_{xy} &= -\frac{\partial}{\partial y}\left[\frac{\partial \varphi_1'}{\partial x} - \frac{\partial \varphi_2'}{\partial x} - x\frac{\partial^2 \varphi_2'}{\partial x^2}\right] \\
2\mu u &= -\frac{\partial \varphi_1'}{\partial x} + x\frac{\partial^2 \varphi_2'}{\partial x} - (1-2\sigma)\frac{\partial \varphi_2'}{\partial x} \\
2\mu v &= -\frac{\partial \varphi_1'}{\partial y} + 2(1-\sigma)\frac{\partial \varphi_2'}{\partial y} + x\frac{\partial^2 \varphi_2'}{\partial x \partial y}.
\end{aligned} \tag{2.4}$$

If we now employ the Cauchy-Riemann equations

$$\frac{\partial \varphi_i'}{\partial x} = \frac{\partial \psi_i'}{\partial y}, \quad \frac{\partial \varphi_i'}{\partial y} = -\frac{\partial \psi_i'}{\partial x}, \quad i = 1, 2 \tag{2.5}$$

to define a conjugate function ψ_i' and note that $\frac{\partial \psi_i'}{\partial x}$ is the function conjugate to $\frac{\partial \varphi_i'}{\partial x}$, we may reduce the order of the derivatives in (2.4), i.e.

$$\tau_{xx} = -\frac{\partial \varphi_1}{\partial x} + x\frac{\partial^2 \varphi_2}{\partial x^2}$$

$$\tau_{yy} = \frac{\partial \varphi_1}{\partial x} - 2\frac{\partial \varphi_2}{\partial x} - x\frac{\partial^2 \varphi_2}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial}{\partial y}\left[\varphi_1 - \varphi_2 - x\frac{\partial \varphi_2}{\partial x}\right]$$

(2.6)
$$2\mu u = -\psi_1 + x\frac{\partial \varphi_2}{\partial x} - (1-2\sigma)\varphi_2$$

$$2\mu v = \psi_1 - 2(1-\sigma)\psi_2 + x\frac{\partial \varphi_2}{\partial y}.$$

Equation (2.6) is obtained from (2.4) by letting $\varphi_i = \frac{\partial \varphi_i'}{\partial x}$ Note that (2.5) defines ψ_i up to an arbitrary constant which corresponds to a rigid body motion.

If either of the stresses or displacements vanishes along the line $x = 0$, this condition may be satisfied identically by choosing the proper relationship between φ_1 and φ_2. This will be illustrated in section 4 and subsequent sections.

3. Orthotropic medium: We treat in this section only the case in which the x and y axes are lines of elastic symmetry of the orthotropic medium. The stresses and displacements may then be defined as follows:

$$\tau_{xx} = \frac{\partial^2 \chi}{\partial y^2}$$

$$\tau_{yy} = \frac{\partial^2 \chi}{\partial x^2}$$

(3.1)
$$\tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} = S_{11}\tau_{xx} + S_{12}\tau_{yy}$$

$$\frac{\partial v}{\partial y} = S_{12}\tau_{xx} + S_{22}\tau_{yy}$$

$$\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = S_{66}\tau_{xy},$$

where

$$(3.2)\qquad \left(\frac{\partial^2}{\partial y^2} + d_1^2 \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2 \chi}{\partial y^2} + d_2^2 \frac{\partial^2 \chi}{\partial x^2}\right) = 0,$$

S_{ij} are the elastic constants in the linear stress-strain law and,

$$(3.3)\qquad d_1^2 d_2^2 = S_{22}/S_{11} \quad , \quad d_1^2 + d_2^2 = 2(S_{12} + S_{66})/S_{11}$$

Now any solution of (3.2) may be expressed as

$$(3.4)\qquad \chi = \Phi_1' + \Phi_2'$$

where

$$(3.5)\qquad \frac{\partial^2 \Phi_i'}{\partial y^2} + d_i^2 \frac{\partial^2 \Phi_i'}{\partial x^2} = 0 \quad , \quad i = 1, 2.$$

The stresses and displacements may then be expressed as:

$$(3.6)\qquad \begin{aligned} \tau_{xx} &= \frac{\partial^2}{\partial y^2}(\Phi_1' + \Phi_2') = -d_1^2 \frac{\partial^2 \Phi_1'}{\partial x^2} - d_2^2 \frac{\partial^2 \Phi_2'}{\partial x^2} \\ \tau_{yy} &= \frac{\partial^2}{\partial x^2}(\Phi_1' + \Phi_2') \\ \tau_{xy} &= -\frac{\partial^2}{\partial x \partial y}(\Phi_1' + \Phi_2') \\ u &= (S_{12} - S_{11} d_1^2)\frac{\partial \Phi_1'}{\partial x} + (S_{12} - S_{11} d_2^2)\frac{\partial \Phi_2'}{\partial x} \\ v &= (S_{12} - S_{11}/d_1^2)\frac{\partial \Phi_1'}{\partial y} + (S_{12} - S_{22}/d_2^2)\frac{\partial \Phi_2'}{\partial y}. \end{aligned}$$

We now define a function Ψ_i' conjugate to Φ_i' as follows:

$$(3.7)\qquad \alpha_i \frac{\partial \Phi_i'}{\partial x} = \frac{\partial \Psi_i'}{\partial y}\,, \quad \frac{\partial \Phi_i'}{\partial y} = -\alpha_i \frac{\partial \Psi_i'}{\partial x}$$

Let

$$(3.8)\qquad \Phi_i = \frac{\partial \Phi_i'}{\partial x}\,, \quad \Psi_i = \frac{\partial \Psi_i'}{\partial x}$$

Then (3.6) reduces to

$$\begin{aligned}
\tau_{xx} &= -\alpha_1^2 \frac{\partial \Phi_1}{\partial x} - \alpha_2^2 \frac{\partial \Phi_2}{\partial x} \\
\tau_{yy} &= \frac{\partial}{\partial x}(\Phi_1 + \Phi_2) \\
(3.9)\qquad \tau_{xy} &= -\frac{\partial}{\partial y}(\Phi_1 + \Phi_2) \\
u &= (S_{12} - S_{11}\alpha_1^2)\Phi_1 + (S_{12} - S_{11}\alpha_2^2)\Phi_2 \\
v &= -\frac{1}{\alpha_1}(\alpha_1^2 S_{12} - S_{22})\Psi_1 - \frac{1}{\alpha_2}(\alpha_2^2 S_{12} - S_{22})\Psi_2 .
\end{aligned}$$

It is obvious from (3.3) that α_1^2 and α_2^2 may be complex. If this is the case the principal value of α_1 is understood throughout.

As in the isotropic case one sees that any one of the stresses or displacements in (3.9) may be made to vanish identically on the line $x = 0$ by the appropriate choice of Φ_1 and Φ_2. Special examples will be given in the following sections. We demonstrate in these sections how the orthotropic problem may be reduced to the corresponding isotropic one.

4. Semi-infinite isotropic medium (loaded along the plane boundary). In this section we assume the plate to occupy the region $X > 0$, which we denote by D. Stresses and/or displacements (or combinations

of the two) are prescribed on the y-axis, the stresses and rotations are continuous throughout D and vanish at infinity. As is well known the displacements may become unbounded at infinity. As was noted in section 2 any one of the stresses or displacements may be made to vanish on $x = 0$ merely by choosing the appropriate relationship between φ_1 and φ_2, e.g. If on $x = 0$ we prescribe

$$(4.1)\qquad \begin{cases} \tau_{xy} = 0 \\ \tau_{xx} = 0 \\ \tau_{yy} = 0 \\ u = 0 \\ v = 0 \end{cases} \quad \text{we choose} \quad \begin{cases} \varphi_1 \equiv \varphi_2 \\ \varphi_1 \equiv 0 \\ \varphi_1 \equiv 2\varphi_2 \\ \varphi_1 \equiv -(1-2\sigma)\varphi_2 \\ \varphi_1 \equiv 2(1-\sigma)\varphi_2 \end{cases}$$

If for instance the line $x = 0$ is shear free as is the case in a number of important physical problems then one obtains the following expressions for the stresses and displacements,

$$(4.2)\qquad \begin{aligned} \tau_{xx} &= -\frac{\partial \varphi}{\partial x} + x\frac{\partial^2 \varphi}{\partial x^2} \\ \tau_{yy} &= -\frac{\partial \varphi}{\partial x} - x\frac{\partial^2 \varphi}{\partial x^2} \\ \tau_{xy} &= x\frac{\partial^2 \varphi}{\partial x \partial y} \\ 2\mu u &= -2(1-\sigma)\varphi + x\frac{\partial \varphi}{\partial x} \\ 2\mu v &= -(1-2\sigma)\psi + x\frac{\partial \varphi}{\partial y} \end{aligned}$$

We note that if we prescribe one of the remaining displacements or stresses on a portion of the shear free line it is equivalent to prescribing the value of a harmonic function or its normal derivative on that portion. Thus if one of the displacements or stresses (or a combination of the two) is prescribed over the entire shear free line one is led to a classical problem in potential theory - a problem in which the value of a harmonic function or its normal derivative (or a combination of the two) is prescribed on the line $x = 0$. It is also apparent that, if any one of the stresses or displacements is prescribed to vanish on $x = 0$ and the value of a displacement, stress, or combination of the two is prescribed along this line, one is led to precisely the same boundary value problem as that arising in the shear free case. Thus in treating the problem in which the y-axis is shear free we treat a representative problem.

We return therefore to the shear free problem and equation (4.2). We note that if we prescribe u over a portion of the shear free boundary it is equivalent to the prescribing of the value of the potential φ on that portion. Similarly the prescription of τ_{xx} over a portion of the y-axis means that the normal derivative of the potential φ is designated there. Two cases are now trivial.

A. If $u = -\frac{(1-\sigma)}{\mu} f(y)$ over the entire y-axis then we have the Fourier integral representation for φ

(a) $f(y)$ an even function of y

$$\varphi = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\cos\alpha y\, d\alpha \int_0^\infty f(y')\cos\alpha y'\, dy'$$

(4.3a)
$$\psi = -\frac{2}{\pi}\int_0^\infty e^{-\alpha x}\sin\alpha y\, d\alpha \int_0^\infty f(y')\sin\alpha y'\, dy'$$

(b) $f(y)$ an odd function of y

$$\varphi = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\sin\alpha y\, d\alpha \int_0^\infty f(y')\sin\alpha y'\, dy'$$

(4.3b)
$$\psi = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\cos\alpha y\, d\alpha \int_0^\infty f(y')\cos\alpha y'\, dy'.$$

B. If $T_{xx} = -g(y)$ for $x = 0$

(a) $g(y)$ even

$$\frac{\partial\varphi}{\partial x} = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\cos\alpha y\, d\alpha \int_0^\infty g(y')\cos\alpha y'\, dy'$$

(4.4)
$$\frac{\partial\psi}{\partial x} = -\frac{2}{\pi}\int_0^\infty e^{-\alpha x}\sin\alpha y\, d\alpha \int_0^\infty g(y')\sin\alpha y'\, dy'$$

(b) $g(y)$ odd

$$\frac{\partial\varphi}{\partial x} = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\sin\alpha y\, d\alpha \int_0^\infty g(y')\sin\alpha y'\, dy'$$

(4.5)
$$\frac{\partial\psi}{\partial x} = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\cos\alpha y\, d\alpha \int_0^\infty g(y')\sin\alpha y'\, dy'$$

If $A(\alpha) = \int_0^\infty g(y')\cos\alpha y'\, dy'$ is $O(\alpha)$ then φ may be obtained in (4.4) by a formal integration with respect to x, the interchange of order of integration being valid. If $B(\alpha) = \int_0^\infty g(y')\sin\alpha y'\, dy'$ is $O(o)$ in α then φ may be obtained

from (4.5) in the same way. A similar statement may be made regarding the determination of ψ .

Problems A and B arise in Boussinesq type problems in the theory of cracks (see Sneddon [27]). If in A we let $\varphi = \psi_1$ and $\psi = -\varphi_1$ equations (4.3a) and (4.3b) will then give solutions to problems in which the displacement v is prescribed on $x = 0$.

Two important cases arise in which the boundary conditions on $x = 0$ are of the mixed type. These are the classical punch and crack problems (see Sneddon [27]). In the punch problem the displacement u is assumed know over a finite interval of the y-axis and the normal stress is assumed to vanish outside this interval. In the crack problem the stress τ_{xx} is prescribed on a portion of the y-axis and u is assumed to vanish outside this interval. Sneddon [26] and Harding and Sneddon [13] have treated a number of such problems by transform methods. This necessitates the introduction of Bessel functions and the solution of a set of dual integral equations. Muskhelishvili [16] Green and Zerna [12] and others have treated such problems by complex variable methods. Although these methods are applicable to a much wider class of problems than those which we shall consider here, they nevertheless seem unduly cumbersome for solving punch and crack problems.

A far more natural way of handling such problems is to introduce an appropriate set of curvilinear coordinates, i.e. if we choose the origin at the center of the finite interval of length

2b on which either u or $\mathcal{T}_{xx}$ is prescribed then in our new coordinate system (ξ, η) the segment $x = 0, -b < y < b$ should be a line ξ = constant while the infinite segments $x = 0, |y| > b$ should be lines on which either ξ or η is constant. (If ξ is constant on these segments the value of the constant must be different from the value of ξ for $-b < y < b$). Two such coordinate systems are well known, namely elliptic coordinates arising from the transformation

$$(4.6) \qquad Z = x + iy = b \sinh \zeta = b \sinh (\xi + i\eta)$$

and bipolar coordinates given by

$$(4.7) \qquad Z = x + iy = -b \cot \tfrac{1}{2} \zeta = -b \cot \tfrac{1}{2} (\xi + i\eta)$$

In terms of elliptic coordinates

$$(4.8) \qquad \begin{aligned} x &= b \sinh \xi \cos \eta , \\ y &= b \cosh \xi \sin \eta . \end{aligned}$$

The semi-infinite region $x > 0$ is given by $-\pi/2 < \eta < \pi/2$, $\xi > 0$. The segment $x = 0, -b < y < b$ corresponds to the interval $-\pi/2 < \eta < \pi/2$ on the η-axis in the ξ, η plane, while the lines $\eta = \pi/2$, $\eta = -\pi/2$ $(\xi > 0)$ in the ξ, η plane are mapped by (4.6) onto the segments, $y > b$, $y < -b$ of the y-axis respectively.

In bipolar coordinates

$$(4.9) \qquad \begin{aligned} x &= -\frac{b \sin \xi}{\cosh \eta - \cos \xi} , \\ y &= \frac{b \sinh \eta}{\cosh \eta - \cos \xi} \end{aligned}$$

The region $X > 0$ is given by $-\pi < \xi < 0$, $-\infty < \eta < \infty$. The finite interval $x = 0, -b < y < b$ corresponds to the line

$\xi = -\pi$ in the ξ,η plane and the intervals $y > b$ and $y < -b$ are given by the intervals $\eta > 0$ and $\eta < 0$ of the η-axis.

C. The punch problem. We assume the displacement u to be prescribed on the interval $-b < y < b$ and take $T_{xx} = 0$ outside this interval on the y-axis. This is equivalent then to the following problem in potential theory: To determine the potential φ regular in any finite open interval $x > 0$ whose derivatives vanish at infinity and which satisfies on $x = 0$ the conditions

$$\varphi = f(y) \quad , \quad -b < y < b$$
$$\frac{\partial \varphi}{\partial x} = 0 \quad , \quad y < -b, \; y > b. \tag{4.10}$$

In elliptic coordinates the boundary conditions are:

$$\varphi = f_1(\eta) \quad , \quad \xi = 0 \; , \; -\frac{\pi}{2} < \eta < \frac{\pi}{2}$$
$$\frac{\partial \varphi}{\partial \eta} = 0 \qquad\qquad \eta = \pm \frac{\pi}{2} \; , \; \xi > 0. \tag{4.11}$$

where $f_1(\eta) = f[y(0,\eta)]$.

If $f(y)$ is an even function of y and can be represented either by the series

$$f(y) = \sum_{n=0}^{\infty} a_n \, b^{-2n} (b^2 - y^2)^n \tag{4.12}$$

or

$$f(y) = \sum_{n=0}^{\infty} c_{2n} \, y^{2n} \tag{4.13}$$

then φ may be obtained in a series of the following type

$$(4.14) \qquad \varphi = \sum_{n=0}^{\infty} A_{2n} e^{-2n\xi} \cos 2n\eta .$$

This is quite easily seen if we note that for $\xi = 0$, $y = b \sin\eta$, and hence $\sqrt{y^2 - b^2} = b \cos\eta$. On the other hand, if $f(y)$ is odd and representable as

$$(4.15) \qquad f(y) = y \sum_{n=0}^{\infty} \alpha_n b^{-2n} (b^2 - y^2)^n$$

or

$$(4.16) \qquad f(y) = \sum_{n=0}^{\infty} \beta_{2n+1} y^{2n+1}$$

then φ may again be readily obtained in series form as

$$(4.17) \qquad \varphi = \sum_{n=0}^{\infty} B_{2n+1} e^{-(2n+1)\xi} \sin (2n+1)\eta .$$

It has been pointed out by a number of authors (see for instance Sneddon [27]) that if the elastic medium remains in contact with the punch over the entire interval $-b < y < b$ then infinite stresses will be produced at the edges of contact. One may remove this infinity by adding to the solution (4.14) a function $A\xi$ (where the constant A must be appropriately determined) or to (4.17) the function $B\eta\xi$. The constant B is again appropriately determined.

As an example let us consider the case in which $f(y)$ is even in y and represented by (4.12). Then for $\xi = 0$ we have

$$(4.18) \qquad \sum_{n=0}^{\infty} A_{2n} \cos 2n\eta = \sum_{m=0}^{\infty} a_m \cos^{2m}\eta .$$

By the classical methods of Fourier analysis we find that

$$A_0 = \frac{1}{2} \sum_{m=0}^{\infty} a_{2m} \frac{\Gamma(2m+1)}{2^{2m-1}\Gamma^2(m+1)}$$

(4.19)

$$A_{2n} = \sum_{m=0}^{\infty} a_{2m} \frac{\Gamma(2m+1)}{2^{2m-1}\Gamma(m+n+1)\Gamma(m-n+1)}$$

If we insert these values back in (4.14) and perform the obviously admissable interchange of order of summation we obtain for φ

$$\varphi = \sum_{m=0}^{\infty} {\sum_{n=0}^{m}}' a_{2m} \frac{\Gamma(2m+1)\, e^{-2n\xi} \cos 2n\eta}{2^{2m-1}\Gamma(m+n+1)\Gamma(m-n+1)}, \tag{4.20}$$

Where the prime indicates that the term n = 0 is to be multiplied by the factor $\frac{1}{2}$. (Since the term n = 0 anyway corresponds to a rigid body motion the prime may be omitted). Now for $\xi = 0$ one has $\frac{\partial \varphi}{\partial x} = \frac{1}{b\cos\eta} \frac{\partial \varphi}{\partial \xi}$. It is obvious then that $\frac{\partial \varphi}{\partial x}$ becomes infinite as $\eta \to \pi/2$. To remove this infinity one adds a multiple of ξ as previously mentioned. It is easily shown that the new function $\varphi' = \varphi + A\xi$ is given by

$$\varphi' = \sum_{m=0}^{\infty} {\sum_{n=0}^{m}}' \frac{a_{2m}\Gamma(2m+1)\left[e^{-2n\xi}\cos 2n\eta + 2(-1)^n n\xi\right]}{2^{2m-1}\Gamma(m+n+1)\Gamma(m-n+1)} \tag{4.21}$$

If $f(y)$ is odd and given by ~~(30)~~ (4.16) then $\varphi' = \varphi + B\eta\xi$ is easily shown to be

$$\varphi' = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \beta_{2m+1} \binom{2m+1}{m-n} \frac{(2n+1)}{2^{2m}} \left[(-1)^n e^{-(2n+1)\xi} \sin(2n+1)\eta + \frac{2}{\pi}\eta\xi\right]. \tag{4.22}$$

On the other hand if $f(y)$ is even in y and vanishes at $y = b$, φ may be represented in bipolar coordinates by the Fourier integral

$$(4.23) \qquad \varphi = \frac{2}{\pi} \int_0^\infty \frac{\cosh \alpha \xi \cos \alpha \eta}{\cosh \alpha \pi} d\alpha \int_0^\infty f_1(\eta') \cos \alpha \eta' d\eta'.$$

In order to eliminate the infinite stress which arises at $y = \pm b$ we form the function

$$(4.24) \qquad \varphi' = \varphi + A \ln \frac{(\cosh \eta/2 + \cos \xi/2)}{(\cosh \eta/2 - \cos \xi/2)}$$

and choose the constant A appropriately. If $f(y)$ is an odd function of y which vanishes for $y = \pm b$ then $\varphi' = \varphi +$

$+ \frac{2}{\pi} B \int_0^\infty \frac{\cosh \alpha \xi \sin \alpha \eta}{\alpha \cosh \alpha \pi} d\alpha$ is given by the Fourier integral

$$(4.25) \qquad \varphi = \frac{2}{\pi} \int_0^\infty \frac{\cosh \alpha \xi \sin \alpha \eta}{\cosh \alpha \pi} \left[\int_0^\infty f'(\eta) \sin \alpha \eta' + \frac{B}{\alpha} \right] d\alpha$$

where B is chosen in such a way that τ_{xx} vanish at $x = 0$, $y = \pm b$. As is peculiar to such problems in two dimensions the displacements become infinite at infinity in the x,y plane, but the stresses and rotations remain finite. Let us illustrate by simple examples the usefulness of the introduction of curvilinear coordinates. We shall treat the cases of the parabolic punch and the elliptical indentation. In the first case for $x = 0$

$$(4.26) \qquad \varphi = k_1 (b^2 - y^2) \quad , -b < y < b \quad , \quad k = \text{const.}$$

$$\frac{\partial \varphi}{\partial x} = 0 \quad , -b > y, \; b < y$$

The potential φ may be represented either by (4.21) or (4.24). In this particular case theelliptic coordinate system is preferable. We obtain at once from (4.21)

$$\varphi = k_1 b^2 \sum_{m=0}^{\infty}{}' \sum_{n=0}^{m}{}' \frac{a_{2m}\,\Gamma(2m+1)}{2^{2m-1}\,\Gamma(m+n+1)\,\Gamma(m-n+1)} \left[e^{-2n\xi} \cos 2n\eta + 2(-1)^n n\xi \right]$$

$$= \frac{k_1 b^2}{2} \left[1 + e^{-2\xi} \cos 2\eta - 2\xi \right] \tag{4.27}$$

from which we find

$$\psi = -\frac{k_1 b^2}{2} \left[e^{-2\xi} \sin 2\eta + 2\eta \right]. \tag{4.28}$$

The second example which we shall give is not a punch problem, but rather a problem in which the displacement u is prescribed, ie.

$$\varphi = k_2 \sqrt{b^2 - y^2} \quad , -b < y < b \quad , \quad k_2 = \text{const.}$$
$$\frac{\partial \varphi}{\partial x} = 0 \quad , y < -b,\ y > b \tag{4.29}$$

Note that for the corresponding elliptic punch problem the punch would not remain in contact with the elastic medium throughout the entire interval $-b < y < b$ but rather in a sub-interval $-a < y < a$, $|a| < |b|$. The solution to the problem under conditions (4.29) may be represented as a doubly infinite series in elliptical coordinates; however, one may obtain it quite easily by employing dipolar coordinates. One notes that for $x = 0$ ($|y| < b$)

$\sqrt{b^2 - y^2} = b \operatorname{sech} \eta/2$ and obtains at once

$$\varphi = \frac{2k_2 b}{\pi}\int_0^\infty \frac{\cosh\alpha\xi}{\cosh\alpha\pi}\cos\alpha\eta\, d\alpha \int_0^\infty \frac{\cos\alpha\eta'}{\cosh\eta'/2}\, d\eta'$$

$$= 2k_2 b \int_0^\infty \frac{\cosh\alpha\xi \cos\alpha\eta}{\cosh^2\alpha\pi}$$

(4.30)

$$= 2\frac{k_2 b}{\pi}\left[\frac{\xi \sin\xi/2 \cosh\eta/2 + \eta\cos\xi/2 \sinh\eta/2}{\cosh\eta - \cos\xi}\right]$$

For $-b < y < b$, $x = 0$

(4.31) $$\tau_{xx} = \frac{2k_2 b}{\pi}\left[\cosh\frac{\eta}{2} - \frac{\eta}{2}\sinh\eta/2\right] = \frac{2k_2 b}{\pi\sqrt{b^2-y^2}}\left[b - y\tanh^{-1}\left(\frac{y}{b}\right)\right]$$

and for $-b > y$, $b < y$, $x = 0$

(4.32) $$u = -\frac{\mu k_2 b}{\pi(1-\sigma)}\frac{\eta}{\sinh\eta/2} = \frac{-2\mu k_2\sqrt{y^2-b^2}}{\pi(1-\sigma)}\coth^{-1}(y/b)$$

We note in passing that for $y > b$ and $f(y)$ even

(4.33) $$-\psi(0,-y) = \psi(0,y) = \psi(0,b)$$
$$= \int_0^b \frac{\partial\psi}{\partial y}\, dy$$
$$= \int_0^b \frac{\partial\varphi}{\partial x}\, dx$$
$$= -P/2$$

where P is the pressure exerted by the punch. If $f(y)$ is an odd function of y we find that

(4.34) $$\psi(0,-y) = \psi(0,y) = \psi(0,b) = P = 0.$$

D. The crack problem. For a formulation and physical interpretation of the crack problem one is referred to the literature (see for instance Sneddon [27] and the references quoted here). In this problem the pressure is prescribed on a finite interval $-b < y < b$ of the y-axis and the displacement u is assumed to vanish outside this interval, i.e. for $x = 0$

$$\frac{\partial \varphi}{\partial x} = g(y) \quad , \quad -b < y < b$$

$$\varphi = 0 \quad , \quad y < -b, \; y > b. \tag{4.35}$$

If g(y) is an even function of y that is representable by (4.12) or (4.13) then an expression of the following type is obtained for φ in elliptic coordinates.

$$\varphi = \sum_{n=0}^{\infty} C_{2n+1} e^{-(2n+1)\xi} \cos(2n+1)\eta. \tag{4.36}$$

If on the other hand $g(y)$ is an odd function of y given by (4.15) or (4.16) φ may be represented as

$$\varphi = \sum_{n=0}^{\infty} D_{2n} e^{-2n\xi} \sin 2n\eta. \tag{4.37}$$

As an example consider the case in which $g(y) = k$ (a constant). Then for $x = 0$, $-b < y < b$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{b \cos\eta} \frac{\partial \varphi}{\partial \xi}\Big|_{\xi = 0} \tag{4.38}$$

Conditions (4.35) may be rewritten for $g(y) = k$, as

$$\frac{\partial \varphi}{\partial \xi} = -k b \cos\eta \quad , \quad \xi = 0$$

$$\varphi = 0 \quad , \quad \eta = \pm \pi/2. \tag{4.39}$$

The obvious solution is

$$\varphi = -kbe^{-\xi}\cos\eta. \tag{4.40}$$

If $g(y)$ is even in y, φ is expressible in dipolar coordinates as

$$\varphi = \frac{b}{\pi}\int_0^\infty \frac{\sinh\alpha\xi\,\cos\alpha\eta\,d\alpha}{\alpha\,\cosh\alpha\pi}\int_0^\infty \frac{g_1(\eta')\cos\alpha\eta'\,d\eta'}{\cosh^2\eta'/2} \tag{4.41}$$

provided $g_1(\eta)$ is $O(e^{\eta-\varepsilon})$ for $\varepsilon>0$ as $\eta\to\infty$. If $g(y)$ is odd then

$$\varphi = \frac{b}{\pi}\int_0^\infty \frac{\sinh\alpha\xi\,\sin\alpha\eta\,d\alpha}{\alpha\,\cosh\alpha\pi}\int_0^\infty \frac{g_1(\eta')\sin\alpha\eta'\,d\eta'}{\cosh^2\eta'/2} \tag{4.42}$$

As a simple example of a problem easily solvable in dipolar coordinates consider the case in which

$$g(y) = b/\sqrt{b^2-y^2} \tag{4.43}$$

Then

$$\varphi = \frac{b}{\pi}\int_0^\infty \frac{\sinh\alpha\xi\,\cos\alpha\eta\,d\alpha}{\alpha\,\cosh\alpha\pi}\int_0^\infty \frac{\cos\alpha\eta'\,d\eta'}{\cosh\eta'/2}$$

$$= b\int_0^\infty \frac{\sinh\alpha\xi\,\cos\alpha\eta\,d\alpha}{\alpha\,\cosh^2\alpha\pi} \tag{4.44}$$

For $x=0$, $(y)>b$

$$\tau_{xx} = -\frac{1}{4\pi}\,\eta \sinh \eta/2$$

(4.45)

$$= -\frac{b}{2\pi}\,\frac{\coth^{-1}(y/b)}{\sqrt{y^2-b^2}}$$

We have listed in this section several examples of plane problems for a semi-infinite region with a shear free plane boundary. As was noted at the beginning of this section numerous other problems can be reduced to boundary value problems of the same type.

5. Semi-infinite orthotropic mediums. We shall demonstrate in this section how orthotropic problems of the type considered in section 4 may be reduced to the corresponding isotropic ones. It is not difficult to show again that any stress or displacement may be made to vanish for $x = 0$ by choosing the appropriate relationship between Φ_1 and Φ_2. As before we take as a typical example the case $\tau_{xy} = 0$ for $x = 0$. We then choose

(5.1) $$\Phi_1(x,y) = \varphi(x_1,y) \quad ; \quad \Phi_2(x,y) = -\varphi(x_2,y)$$

where

(5.2) $$X_i = \alpha_i^{-1} X$$

and

(5.3) $$\frac{\partial^2\varphi}{\partial x_i^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

We have then for $x = 0$

$$u = s_{11}(\alpha_2^2 - \alpha_1^2)\varphi(0,y)$$

$$\tau_{xx} = (\alpha_2 - \alpha_1)\frac{\partial \varphi}{\partial x_1}(x_1,y)\Big|_{x_1=0}$$

(5.4)
$$v = -\left[\frac{1}{\alpha_1}(\alpha_1^2 s_{12} - s_{22}) + \frac{1}{\alpha_2}(\alpha_2^2 s_{12} - s_{22})\right]\psi(0,y)$$

$$\tau_{yy} = \frac{\alpha_2 - \alpha_1}{\alpha_1 \alpha_2}\frac{\partial \varphi}{\partial x_1}(x_1,y)\Big|_{x_1=0}$$

$$\tau_{xy} = 0,$$

and the crack and punch problems reduce to the same problems in potential theory that were encountered in the previous section. The function φ is a harmonic function satisfying the same type of boundary conditions as were required in the corresponding isotropic problem. Similarly if for $x = 0$

(5.5)
$$\begin{cases} \tau_{xx} = 0 \\ \tau_{yy} = 0 \\ \tau_{xy} = 0 \\ u = 0 \\ v = 0 \end{cases}$$
we choose
$$\begin{cases} \Phi_1 = \alpha_2 \varphi(x_1,y), \ \Phi_2 = -\alpha_1 \varphi(x_2,y) \\ \Phi_1 = \alpha_1 \varphi(x_1,y), \ \Phi_2 = -\alpha_2 \varphi(x_2,y) \\ \Phi_1 = \varphi(x_1,y), \ \Phi_2 = -\varphi(x_2,y) \\ \Psi_1 = (s_{12} - s_{11}\alpha_2^2)\varphi(x_1,y), \ \Psi_2 = -(s_{12} - s_{11}\alpha_1^2)\varphi(x_2,y) \\ \Psi_1 = \alpha_1(\alpha_2^2 s_{12} - s_{22})\psi(x_1,y), \ \Psi_2 = -\alpha_2(\alpha_1^2 s_{12} - s_{22})\psi(x_2,y) \end{cases}$$

It is obvious that in each case the orthotropic problem is reduced to the corresponding isotropic one.

6. Isolated singularities in a semi-infinite medium. We suppose that an isolated singularity acts at a point (c,o) in a semi-infinite isotropic or orthotropic medium. Suppose further that one of the stresses or displacements vanishes on the boundary $x = 0$. Since the boundary value problem is of the same type

regardless of which stress or displacement is prescribed we take as a typical example the case $\tau_{xy} = 0$ on the y-axis. We then express the stress distribution as the superposition of two stress systems, i.e.

$$\begin{aligned} \tau_{xx} &= \tau_{xx}^{1} + \tau_{xx}^{2} \\ \tau_{xy} &= \tau_{xy}^{1} + \tau_{xy}^{2} \\ u &= u^{1} + u^{2} \text{ , etc.} \end{aligned} \tag{6.1}$$

The distribution of stress corresponding to superscript 1 will be that due to the combination of two isolated singularities acting at (c,o) and the image point (-c,o) in an infinite medium. The singularities will be of equal strength, but of equal or opposite sign depending on which stress or displacement was originally prescribed to vanish on x = 0. For $T_{xy} = 0$ the singularities are chosen of opposite sign so that $T_{xy}^{1} = 0$. We may now represent the stress system with superscript 2 by (4.2) if the region is isotropic and by (3.9) and (5.1) if the region is orthotropic. Since the stresses and displacements with superscript 1 are known the problem of determining the system with superscript 2 is precisely the problem treated in section 4(or section 5 if the medium is orthotropic). We now give some examples.

We note first that the Airy functions χ_1 and χ_2 for isolated singularities at the point (c,o) in the x and y directions (isotropic medium) are

$$\text{(a)}\quad \chi = \frac{P}{4\pi(1-\sigma)}\left[(1-2\sigma)(x-c)\ln\sqrt{(x-c)^2+y^2} - 2(1-\sigma)\,y\tan^{-1}y/x-c\right] \tag{6.2}$$

$$\text{(b)}\quad \chi = \frac{P}{4\pi(1-\sigma)}\left[(1-2\sigma)\,y\ln\sqrt{(x-c)^2+y^2} - 2(1-\sigma)(x-c)\tan^{-1}(x-c)/y\right].$$

For an isolated force in the x-direction we find

$$2\mu u = \frac{P}{4\pi(1-\sigma)}\left[\frac{(x-c)^2}{r_1^2} - \ln r_1\right]$$

$$2\mu v = \frac{P}{4\pi(1-\sigma)}\left[\frac{(x-c)y}{r_1^2}\right]$$

$$\tau_{yy} = \frac{P}{4\pi(1-\sigma)}\left[\frac{(1-2\sigma)(x-c)}{r_1^2} - \frac{2(x-c)y^2}{r_1^4}\right]$$

(6.3)

$$\tau_{xx} = \frac{P}{4\pi(1-\sigma)}\left[-\frac{(3-2\sigma)(x-c)}{r_1^2} + \frac{2(x-c)y^2}{r_1^4}\right]$$

$$\tau_{xy} = \frac{P}{4\pi(1-\sigma)}\left[(1-2\sigma)\frac{y}{r_1^2} + \frac{2(x-c)^2 y}{r_1^4}\right]$$

Where $r_1^2 = (x-c)^2 + y^2$,

with similar expressions for loading in the y direction, obtained by interchanging $(x - c)$ and y. It is clear then that if the effects of an isolated force of the same magnitude and sign at the image point $(-c, 0)$ are superposed on the original stress system, then for $x = 0$, $T^1_{xx} = T^1_{yy} = v^1 = 0$. If the forces are of the same magnitude but opposite in sign then $T^1_{xy} = u^1 = 0$. As we mentioned before we shall treat specifically the latter case in which the boundary $x = 0$ is to be shear free. Then for $x = 0$

$$2\mu u^1 = 0$$

$$2\mu v^1 = -\frac{Pcy}{\pi(1-\sigma)[y^2+c^2]}$$

(6.4)

$$\tau^1_{yy} = \frac{P}{2\pi(1-\sigma)}\left[-\frac{(1-2\sigma)c}{y^2+c^2} + \frac{2cy^2}{(y^2+c^2)^2}\right]$$

$$\tau^1_{xx} = \frac{P}{2\pi(1-\sigma)}\left[\frac{(1-2\sigma)c}{y^2+c^2} + \frac{2c^3}{(y^2+c^2)^2}\right]$$

$$\tau^1_{xy} = 0.$$

The problem of specifying any of the displacements or stresses on the shear free boundary ($x = 0$) is now reduced to the problem considered in section 2. For instance if the additional boundary conditions are $u = 0$, $|y| > b$; $\tau_{xx} = 0$, $|y| < b$, (the problem of an unloaded crack) then by (4.2) one is required to solve the following crack problem.

$$\varphi = 0 \qquad |y| > b$$

$$\frac{\partial \varphi}{\partial x} = \frac{P}{2\pi(1-\sigma)}\left[\frac{(1-2\sigma)c}{(y^2+c^2)} + \frac{2c^3}{(y^2+c^2)^2}\right], \quad |y| < b. \tag{6.5}$$

The solution is given formally by the Fourier integral

$$\varphi = -\frac{P}{2\pi^2(1-\sigma)}\int_0^\infty \frac{\sinh\alpha\zeta\cos\alpha\eta}{\alpha\cosh\alpha\pi}\,d\alpha \int_0^\infty\left[\frac{(1-2\sigma)c}{(b^2\tanh^2\frac{\alpha}{2}+c^2)} + \frac{2c^3}{(b^2\tanh^2\frac{\alpha}{2}+c^2)^2}\right]\frac{\cos\alpha\eta'}{\cosh\frac{\alpha}{2}}\,d\eta' \tag{6.6}$$

which can perhaps best be evaluated numerically in a specific case. The complete solution is obtained according to (6.1) by adding to (6.3) the solution for the image problem and the results obtained by inserting (6.6) in (4.2). If $x = 0$ is to be stress free we must add a solution of (4.2) which on $x = 0$ will cancel the τ^1_{xx} of (6.4). We choose in this case

$$\varphi = \frac{P}{2\pi(1-\sigma)}\left\{A\ln\left[(x+c)^2+y^2\right] + \frac{Bc(x+c)}{(x+c)^2+y^2}\right\} \tag{6.7}$$

Then

$$\tau_{xx}\Big|_{x=0} = \frac{P}{2\pi(1-\sigma)}\left\{\left[\frac{(1-2\sigma)c}{(y^2+c^2)} + \frac{2c^3}{(y^2+c^2)^2}\right] - \left[\frac{2Ac}{y^2+c^2} + \frac{2Bc}{y^2+c^2} - \frac{2Bc^3}{(y^2+c^2)^2}\right]\right\} = 0 \tag{6.8}$$

The condition that (6.8) vanish determines the constants A and B as

$$(6.9) \qquad B = -1 \ , \quad A = \frac{3-2\sigma}{2}$$

The complete stress distribution is then obtained by adding to (6.3) the results for the image load and those obtained by inserting (6.6) in (4.2).

If the y-axis is to be fixed ($u = 0$, $v = 0$) then we must add to the stress distribution due to the two point loads, a solution to (2.6) with $\varphi_1 = -(1-2\sigma)\varphi_2 = -(1-2\sigma)\varphi$, such that the displacement v' on $x = 0$ corresponds to this choice of φ cancels the v' in (6.4). With this choice of φ_1 and φ_2 (2.6) reduces to

$$\begin{aligned} \tau'_{xx} &= (1-2\sigma)\frac{\partial \varphi}{\partial x} + x\frac{\partial^2 \varphi}{\partial x^2} \\ \tau'_{yy} &= -(3-2\sigma)\frac{\partial \varphi}{\partial x} - x\frac{\partial^2 \varphi}{\partial x^2} \\ (6.10) \qquad \tau'_{xy} &= -\frac{\partial}{\partial y}\left[-2(1-\sigma)\varphi - x\frac{\partial \varphi}{\partial x}\right] \\ 2\mu u' &= x\frac{\partial \varphi}{\partial x} \\ 2\mu v' &= -(3-4\sigma)\psi + x\frac{\partial \varphi}{\partial y} \end{aligned}$$

The condition that v vanish for $x = 0$ requires that (from 6.4 and 6.10)

$$(6.11) \qquad -\frac{Pcy}{\pi(1-\sigma)(y^2+c^2)} - (3-4\sigma)\psi = 0 \ , \quad x = 0$$

It is clear then that

$$(6.12) \qquad \psi = -\frac{Pc}{\pi(1-\sigma)(3-4\sigma)} \frac{y}{[(x+c)^2+y^2]}$$

$$\varphi = \frac{Pc}{\pi(1-\sigma)(3-4\sigma)} \frac{x+c}{[(x+c)^2+y^2]} .$$

It can also be easily shown that for an elastically supported bdry., ($T_{xy} = 0$, $T_{xx} - ku = 0$, see [3]) the stress distribution due to a point load at(c,o) is obtained by superposing the effects of a point load at (c,o) in an infinite region, a point load of the same magnitude but of opposite sign at (-c,o) in the infinite region and the stress distribution obtained by inserting the function

$$(6.13) \qquad \varphi = -\frac{P\mu}{2\pi(1-\sigma)} \int_0^\infty e^{-\alpha(x+c)} \frac{[2(1-\sigma)+\alpha c]}{[k(1-\sigma)+\alpha\mu]} \cos\alpha y \, d\alpha,$$

into (4.2).

Suppose one want the stress distribution due to a point load at (c,o) in a semi-infinite region elastically supported along $x = 0$ ($T_{xx} - k u = 0$) and in which the boundary $x = 0$ is constrained in such a way that no lateral motion is possible ($v = 0$). One superposes on (6.3) the stress distribution due to a point load ⊗ at (-c,o) in the infinite region which is of the same magnitude and sign as that at (c,o). The resulting displacement v will vanish on $x = 0$, as will T_{xx}; however,

$$(6.14) \qquad u = \frac{P}{4\pi\mu(1-\sigma)} \left[\frac{c^2}{y^2+c^2} - \ln\sqrt{y^2+c^2}\right]$$

One must then superpose on the system of two isolated forces a stress distribution which is regular for $x > 0$ (displacements will become logarithmically infinite but stresses will vanish at infinity). To effect this one chooses in (2.6)

$\psi_1 = 2(1-\sigma)\psi_2 = 2(1-\sigma)\psi$ (so that v will vanish on y-axis).

It can also be easily shown that for an elastically supported bdry., ($T_{xy} = 0$, $T_{xx} - ku = 0$, see [3]) the stress distribution due to a point load at(c,o) is obtained by superposing the effects of a point load at (c,o) in an infinite region, a point load of the same magnitude but of opposite sign at (-c,o) in the infinite region and the stress distribution obtained by inserting the function

$$(6.13) \qquad \varphi = -\frac{P\mu}{2\pi(1-\sigma)} \int_0^\infty e^{-\alpha(x+c)} \frac{[2(1-\sigma)+\alpha c]}{[k(1-\sigma)+\alpha\mu]} \cos\alpha y \, d\alpha,$$

into (4.2).

Suppose one want the stress distribution due to a point load at (c,o) in a semi-infinite region elastically supported along $x = 0$ ($T_{xx} - k u = 0$) and in which the boundary $x = 0$ is constrained in such a way that no lateral motion is possible ($v = 0$). One superposes on (6.3) the stress distribution due to a point load @ at (-c,o) in the infinite region which is of the same magnitude and sign as that at (c,o). The resulting displacement v will vanish on $x = 0$, as will T_{xx}; however,

$$(6.14) \qquad u = \frac{P}{4\pi\mu(1-\sigma)} \left[\frac{c^2}{y^2+c^2} - \ln\sqrt{y^2+c^2}\right]$$

One must then superpose on the system of two isolated forces a stress distribution which is regular for $x > 0$ (displacements will become logarithmically infinite but stresses will vanish at infinity). To effect this one chooses in (2.6)

$\psi_1 = 2(1-\sigma)\psi_2 = 2(1-\sigma)\psi$ (so that v will vanish on y-axis).

This gives

$$
\begin{aligned}
\tau_{xx}^{2} &= -2(1-\sigma)\frac{\partial \varphi}{\partial x} + x\frac{\partial^2 \varphi}{\partial x^2} \\
\tau_{yy}^{2} &= 2\sigma\frac{\partial \varphi}{\partial x} - x\frac{\partial^2 \varphi}{\partial x^2} \\
\tau_{xy}^{2} &= -\frac{\partial}{\partial y}\left[(1-2\sigma)\varphi - x\frac{\partial \varphi}{\partial x}\right] \\
2\mu u^2 &= -(3-4\sigma)\varphi + x\frac{\partial \varphi}{\partial x} \\
2\mu v^2 &= x\frac{\partial \varphi}{\partial y}.
\end{aligned}
\tag{6.15}
$$

One must then solve for φ such that

$$
-2(1-\sigma)\frac{\partial \varphi}{\partial x} + k(3-4\sigma)\varphi = \frac{Pk}{4\pi\mu(1-\sigma)}\left[\frac{c^2}{y^2+c^2} - \ln\sqrt{y^2+c^2}\right]. \tag{6.16}
$$

Choose $\varphi = -\frac{P \ln\sqrt{(x+c)^2+y^2}}{4\pi\mu(1-\sigma)(3-4\sigma)} + \varphi_3$. Then φ_3 must satisfy

$$
-2(1-\sigma)\frac{\partial \varphi_3}{\partial x} + k(3-4\sigma)\varphi_1 = P_c\frac{[(3-4\sigma)kc - 2(1-\sigma)]}{4\pi\mu(1-\sigma)(3-4\sigma)(y^2+c^2)} \quad (\text{for } x=0) \tag{6.17}
$$

from which one obtains

$$
\varphi_1 = \frac{P_c[(3-4\sigma)kc - 2(1-\sigma)]}{4\pi\mu(1-\sigma)(3-4\sigma)}\int_0^\infty \frac{e^{-\alpha(x+c)}\cos\alpha y\, d\alpha}{[2(1-\sigma)\alpha + k(3-4\sigma)]} \tag{6.18}
$$

One can treat such problems under various other boundary conditions in a similar manner. Likewise the effects of isolated couples, line loads, etc. may be obtained by differentiation and integration, of the solution to the isolated force problem.

If the medium is orthotropic the method of solution indicated in the beginning of this section presents no additional difficulties.

7. Infinite strip problems. In the two preceding sections we treated problems in which the region of definition was the half space $x>0$. We suppose now that the elastic medium occupies the strip, $0<x<d$, $-\infty<y<\infty$. We treat first the case in which stresses are applied at the boundaries $x = 0$, $x = d$. We note in passing that each of the potentials φ_1 and φ_2 in (2.6) represents two independent sets of harmonic functions. In the previous cases the conditions at infinity eliminated one of the two independent sets of functions. In general we shall let

$$\begin{aligned} \varphi_1 &= k_1\varphi + \varphi_1' \\ \varphi_2 &= k_2\varphi + \varphi_2' \end{aligned} \tag{7.1}$$

the k_1, k_2, φ_1' and φ_2' so chosen that φ is the solution to the problem for the semi-infinite region having the same boundary conditions at $x = 0$ as those of the strip problem under consideration. This means that φ_1' and φ_2' must give no contribution to the boundary conditions on $x = 0$. This choice can always be made provided the boundary conditions on $x = 0$ are such that the boundary value problem for φ is not of mixed type. It usually amounts to choosing φ_1' and φ_2' as either even or odd functions of x. In addition, however, φ_1' and φ_2' must be chosen to satisfy the conditions on $x = d$. This choice can again be made provided the resulting boundary value problems for φ_1' and φ_2' are not of mixed type.

Let us take as a simple example a strip problem in which the boundary $x = 0$ is shear free and u is prescribed on the boundaries $x = 0$, $x = d$. We choose then in (2.6),

$$\begin{aligned} \varphi_1 &= \varphi + \varphi_1' \\ \varphi_2 &= \varphi + \varphi_2' \end{aligned} \tag{7.2}$$

Where φ_1' and φ_2' are odd functions of x. Then the expressions for the stresses and displacements become:

$$\begin{aligned} \tau_{xx} &= \left[-\frac{\partial \varphi}{\partial x} + x\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \varphi_1'}{\partial x} + x\frac{\partial^2 \varphi_2'}{\partial x} \right] \\ \tau_{yy} &= \left[-\frac{\partial \varphi}{\partial x} - x\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \varphi_1'}{\partial x} - 2\frac{\partial \varphi_2'}{\partial x} - x\frac{\partial^2 \varphi_2'}{\partial x} \right] \\ \tau_{xy} &= \frac{\partial}{\partial y}\left[x\frac{\partial \varphi}{\partial x} - \varphi_1' + \varphi_2' + x\frac{\partial \varphi_2'}{\partial x} \right] \\ 2\mu u &= \left[-2(1-\sigma)\varphi + x\frac{\partial \varphi}{\partial x} - \varphi_1' - (1-2\sigma)\varphi_2' + x\frac{\partial \varphi_2'}{\partial x} \right] \\ 2\mu v &= \left[-(1-2\sigma)\psi + x\frac{\partial \varphi}{\partial y} + \psi_1' - 2(1-\sigma)\psi_2' + x\frac{\partial \varphi_2'}{\partial y} \right]. \end{aligned} \tag{7.3}$$

Since φ_1' and φ_2' are odd functions of x then on the strip boundaries

$$\begin{aligned} &\tau_{xy} = 0 \quad , \quad x = 0, \; x = d \\ &u = -\frac{(1-\sigma)}{\mu}\varphi = -\frac{(1-\sigma)f(y)}{\mu}, \; x = 0 \; ; \; 2\mu u = g(y) \; , \; x = d. \end{aligned} \tag{7.4}$$

We choose for φ the solution to the corresponding problem for the semi-infinite region, i.e.

$$\varphi = \frac{2}{\pi}\int_0^\infty e^{-\alpha x}\cos\alpha y \, d\alpha \int_0^\infty f(y')\cos\alpha y' \, dy', \tag{7.5}$$

and for φ_1' and φ_2'

$$\varphi_1' = \int_0^\infty A(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha$$

(7.6)

$$\varphi_2' = \int_0^\infty B(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha .$$

The conditions on x = d require that

$$-2(1-\sigma)\varphi + d\frac{\partial \varphi}{\partial x} - \varphi_1' - (1-2\sigma)\varphi_2' + d\frac{\partial \varphi_2'}{\partial x} = g(y)$$

(7.7)

$$d\frac{\partial \varphi}{\partial x} - \varphi_1' + \varphi_2' + d\frac{\partial \varphi_2'}{\partial x} = 0$$

or

(7.8)

$$\varphi_2' = -\varphi - \frac{g(y)}{2(1-\sigma)} \; ; \; \varphi_1' = d\frac{\partial \varphi}{\partial x} + \varphi_2' + d\frac{\partial \varphi_2'}{\partial x}$$

We now write

(7.9)

$$g(y) = \frac{2}{\pi}\int_0^\infty \cos \alpha y \, d\alpha \int_0^\infty g(y') \cos \alpha y' \, dy',$$

from which we find

$$A(\alpha) = \frac{-1}{\sinh \alpha d}\left[\frac{2}{\pi} \alpha d \, e^{-\alpha d}\int_0^\infty f(y') \cos \alpha y' \, dy' + B(\alpha)(1+\alpha d \coth \alpha d)\right]$$

(7.10)

$$B(\alpha) = \frac{-2}{\pi \sinh \alpha d}\left\{\int_0^\infty \left[e^{-\alpha d} f(y') + \frac{g(y')}{2(1-\sigma)}\right] \cos \alpha y' \, dy'\right\}.$$

We list now convenient choices for φ_1 and φ_2 for various boundary conditions on x = 0. The superscript zero on φ will stand for an even function of x and the superscript one for an odd function.

A. $u = 0, \quad v = f_1(y)$

(a) $\varphi_1 = -(1-2\sigma)(\varphi + \varphi_1^{\circ}) + 2(1-\sigma)\varphi_2'$

(b) $\varphi_2 = \varphi + \varphi_1^{\circ} + \varphi_2'$

B. $T_{xy} = 0, \quad T_{xx} = f_2(y)$

(a) $\varphi_1 = \varphi + \varphi_1^{\circ}$

(b) $\varphi_2 = \varphi + \varphi_1^{\circ} + \varphi_2'$

C. $T_{xx} = 0, \quad T_{xy} = f_3(y)$

(a) $\varphi_1 = \varphi_1^{\circ}$

(b) $\varphi_2 = \varphi + \varphi_1^{\circ} + \varphi_2'$

D. $v = 0, \quad T_{xy} = f_4(y)$

(a) $\varphi_1 = \varphi + \varphi_1' + \varphi_2^{\circ}$

(b) $\varphi_2 = 2(1-\sigma)\varphi + \varphi_1' + \varphi_2^{\circ}$,

etc.

If an isolated load is applied at a point (c,o), $0 < c < d$ in an infinite strip, the procedure for handling such a problem is a combination of the methods proposed in 6 and in the first part of this section. We express the resultant stresses and displacements as in (6.1). The stress system with superscript 1 is precisely the same as before, i.e. the distribution due to the combination of an isolated force at (c,o) and another at (-c,o) of equal magnitude but opposite sign (if the y-axis is shear free). We then represent the system with superscript 2 by (7.2) and proceed as indicated earlier in this section. The conditions under which the problem is solvable are precisely those mentioned previously in this section.

Orthotropic strip problems are treated in precisely the same manner. As an example suppose $T_{xy} = 0$ and u is prescribed on the two boundaries $x = 0$, $x = d$. Then one chooses for Φ_1 and Φ_2 in (3.9).

$$\Phi_1 = \varphi(x_1, y) + \Phi_1'(x, y)$$
$$\Phi_2 = \varphi(x_2, y) + \Phi_2'(x, y), \tag{7.11}$$

where $\Phi_1'(x,y)$ and $\Phi_2'(x,y)$ are odd functions of x and φ satisfies (5.3). The potential φ is in fact the solution to the half plane problem satisfying the same boundary conditions on $x = 0$, and if $u = h(y)$ on $x = d$, $\Phi_1'(x,y)$ and $\Phi_2'(x,y)$ are easily determined from this condition, i.e.

$$\Phi_2'(d, y) = -\varphi(\alpha_2^{-1}d, y) + \frac{h(y)}{S_{11}(\alpha_1^2 - \alpha_2^2)},$$
$$\Phi_1'(d, y) = -\varphi(\alpha_1^{-1}d, y) - \frac{h(y)}{S_{11}(\alpha_1^2 - \alpha_2^2)}. \tag{7.12}$$

8. <u>Holes or cutouts in semi-infinite plates:</u> If stresses and/or displacements are prescribed on an internal boundary of a semi-infinite medium the Airy's stress function approach still often gives perhaps the most convenient method for attacking the problem. In this case one may prefer to use a slightly different procedure than that previously discussed in this report. One has the following admissable decompositions of the Airy function χ.

a) $\chi = \varphi_1 + x\varphi_2$ (Almansi [2])

(8.1) b) $\chi = \varphi_1 + \varphi_2^{-2}$ (Weinstein [32])

where φ_2^{-2} satisfies the differential equation

(8.2) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{2}{x}\frac{\partial}{\partial x}\right)\varphi_2^{-2} = 0$$

and may be interpreted in the language of Generalized Axially Symmetric Potential Theory as a stream function in four space. The φ_1 and φ_2 in (8.1 a) and(8.1 b) each represent two independent solutions. We now make use of the correspondence relation (see Weinstein [33])

(8.3) $$\varphi^{-k} = x^{-k+1}\varphi^{k+2}$$

where

(8.4) $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{(k+2)}{x}\frac{\partial}{\partial x}\right]\varphi^{k+2} = 0$$

Equation (8.3) may be used to express any function φ^{-2} in terms of an axially symmetric potential φ^{k} in "6 space" and vice versa. With little difficulty one can show that χ may be decomposed in the following way:

(8.5) $$\chi = \varphi_1^{\circ} - x\frac{\partial \varphi_2^{\circ}}{\partial x} + x^2\left[\frac{\partial \varphi_1^{2}}{\partial x} + \frac{\partial \varphi_2^{2}}{\partial y}\right]$$

where we now use the notation φ^{k} to signify that

(8.6) $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{k}{x}\frac{\partial}{\partial x}\right]\varphi^{k} = 0.$$

It is now apparent that if $\tau_{xy} = \tau_{xx} = 0$ for $x = 0$, φ_1° and φ_2° in (8.5) are identically zero. If $u = 0$, $\tau_{xy} = 0$, $\varphi_2^{\circ} \equiv \varphi_1^{\circ} \equiv 0$ and φ_1° is even in x, etc.

Let us consider now the problem of determining the stress distribution in a semi-infinite medium when stresses are applied to the boundary of a circular hole and the plane boundary is stress free. One introduces dipolar coordinates ($x + iy = ib \cot\frac{1}{2}\zeta$) in terms of which χ is given by

$$\chi = (\cosh\eta - \cos\xi)^{-1} \sum_{n=1}^{\infty} \left\{ A_n \sinh\eta \sinh n\eta + B_n [n \sinh\eta \cosh n\eta - \cosh\eta \sinh n\eta] \right\} \left\{ C_n \cos n\xi + D_n \sin n\xi \right\}, \tag{8.7}$$

the coefficients being determined by the nature of the distribution of stress on the circle $\eta = \eta_0$.

A similar treatment will yield the solution when the semi-infinite medium possesses a crack in the shape of an arc of a circle or when the boundary $x = 0$ is notched in various ways.

9. Infinite strip problems with mixed boundary conditions: The author is unaware of any method which will yield the solution to the problem of the infinite elastic strip when the boundary conditions are of mixed type. However, approximate solutions to special problems of this type can be obtained. For instance if only one of the boundary conditions on only one of the boundaries is of mixed

type the solution may be approximated by an iterative procedure in one of two ways. In the first case the final result depends on the solution to a set of dual integral equations of the type

$$\int_0^\infty G(\alpha) f(\alpha) \begin{Bmatrix} \sin \alpha y \\ \cos \alpha y \end{Bmatrix} d\alpha = g(y) \qquad (0<|y|<b)$$

$$\int_0^\infty f(\alpha) \begin{Bmatrix} \sin \alpha y \\ \cos \alpha y \end{Bmatrix} d\alpha = 0 \qquad (|y|>b) \tag{9.1}$$

where $\sin \alpha y$ applies for the odd part of $g(y)$ and $\cos \alpha y$ for the even part. An iterative procedure for handling such a set of dual integral equations has been given by Tranter [31]. Alternatively one may express the solution as a superposition of various stress distribution. The first is the solution to the half plane problem with the same conditions on $x = 0$ (We assume the mixed boundary condition to be satisfied here). We then choose two additional solutions which satisfy the conditions on $x = d$ and the non-mixed condition on $x = 0$. (This amounts to choosing even or odd functions for φ_1' and φ_1^2). However, the mixed boundary condition is no longer satisfied. Hence we proceed to solve the new mixed boundary value problem in the same manner. The question of the convergency of these procedures has not been investigated except for very special cases. Let us show by an example how each would be applied. Consider the following boundary value problem: To determine the distribution of stress in an infinite elastic strip under the following conditions:

$$(9.2)\qquad \begin{aligned} &(a)\quad \tau_{xy} = 0 \quad , \quad x = \infty \\ &\qquad \tau_{xx} = k \quad , \ |y| < b \ ; \quad u = 0 \quad |y| > b \quad , \quad x = 0 \\ &(b)\quad u = 0 \quad , \quad x = d \\ &\qquad \tau_{xy} = 0 \quad , \quad x = d \end{aligned}$$

Equations (7.3) give the appropriate expressions for the stresses and displacements. In the first case we assume

$$(9.3)\qquad \begin{aligned} \varphi &= \int_0^\infty A(\alpha)\, e^{-\alpha x} \cos \alpha y \, d\alpha \\ \varphi_1' &= \int_0^\infty B(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha \\ \varphi_2' &= \int_0^\infty C(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha . \end{aligned}$$

The last two boundary conditions require that

$$(9.4)\qquad B(\alpha) = -\left[\frac{e^{-\alpha d}}{\sinh \alpha d}\left(1 + \alpha d + \alpha d \coth \alpha d\right)\right] A(\alpha)$$

Then for $x = 0$ the second of boundary conditions (9.2a) requires that

$$(9.5)\qquad \begin{aligned} &(a)\quad \int_0^\infty \frac{\alpha A(\alpha)}{\sinh \alpha d}\left[\sinh \alpha d + e^{-\alpha d}\left(1 + \alpha d + \alpha d \coth \alpha d\right)\right] \cos \alpha y \, d\alpha = k, \quad |y| < b \\ &(b)\quad \int_0^\infty A(\alpha) \cos \alpha y \, d\alpha = 0 \quad , \quad |y| > b , \end{aligned}$$

which are integrals of type (9.1). The alternative procedure would be to choose in (7.3)

$$(a) \qquad \varphi = -kb \int_0^\infty \frac{e^{-\alpha x} \cos \alpha y \, J_1(\alpha b)\, d\alpha}{\alpha}$$

$$(9.6) \quad (b) \qquad \varphi_1' = \int_0^\infty B_1(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha$$

$$(c) \qquad \varphi_2' = \int_0^\infty C_1(\alpha) \sinh \alpha x \cos \alpha y \, d\alpha$$

The functions $B_1(\alpha)$ and $C_1(\alpha)$ are so chosen that the conditions on $x = d$ are satisfied. This gives for $B_1(\alpha)$

$$(9.7) \qquad B_1(\alpha) = kb \left[\frac{e^{-\alpha d} J_1(\alpha b)}{\alpha \sinh \alpha d} (1 + \alpha d + \alpha d \coth \alpha d) \right].$$

But the mixed condition on $x = 0$ is no longer satisfied. Instead we have from (7.3)

$$(9.8) \qquad \tau_{xx} = k - kb \int_0^\infty \frac{e^{-\alpha d} J_1(\alpha b)}{\sinh \alpha d} (1 + \alpha d + \alpha d \coth \alpha d) \cos \alpha y \, d\alpha \,, \quad |y| < b$$

$$u = 0 \,, \qquad |y| > b$$

Formally at least this procedure may be repeated for the new mixed boundary value problem (9.8). Practically one is confronted in each procedure with integrals which must be approximated or integrated numerically. There is also of course the question of convergence of the iterative procedure. From purely physical reason one would expect rather rapid convergence in each case provided d is sufficiently large.

It should perhaps be pointed out that one need not use transform methods in determining φ in the second procedure. The solution φ may be obtained in curvilinear coordinates from whence the value of φ along the y-axis is calculated and expressed as a function of y. From the knowledge of φ along the entire y-axis one can immediately write down the expression for φ in the half-plane $x > 0$ in x,y coordinates.

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